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Universal Lax pairs for spin Calogero–Moser models and spin exchange models

V I Inozemtsev[1] **and R Sasaki**

Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

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Abstract
For any root system Δ and a set of vectors $\mathcal{R}$ which form a single orbit of the reflection (Weyl) group G_Δ generated by Δ, a *spin Calogero–Moser model* can be defined for each of the potentials: rational, hyperbolic, trigonometric and elliptic. For each member μ of $\mathcal{R}$, to be called a 'site', we associate a vector space $\mathbf{V}_\mu$ whose element is called a 'spin'. Its dynamical variables are the canonical coordinates $\{q_j, p_j\}$ of a particle in $\mathbf{R}^r$ (r = rank of Δ) and spin exchange operators $\{\hat{\mathcal{P}}_\rho\}$ ($\rho \in \Delta$) which exchange the spins at the sites μ and $s_\rho(\mu)$. Here s_ρ is the reflection generated by ρ. For each Δ and $\mathcal{R}$ a *spin exchange model* can be defined. The Hamiltonian of a spin exchange model is a linear combination of the spin exchange operators only. It is obtained by 'freezing' the canonical variables at the equilibrium point of the corresponding classical Calogero–Moser model. For $\Delta = A_r$ and $\mathcal{R}$ = set of vector weights it reduces to the well-known Haldane–Shastry model. Universal Lax pair operators for both spin Calogero–Moser models and spin exchange models are presented which enable us to construct as many conserved quantities as the number of sites for *degenerate* potentials.

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1. Introduction

The essential part of our knowledge of quantum many-body systems is concerned with integrable models in one dimension. Among other well-known theories such as the sine-Gordon and affine Toda field theories, correlated electron models (Hubbard, *t–J*, etc) and quantum spin chains (*XXZ*, etc), the Calogero–Moser (CM) models [1–4] with long-range interactions are actively investigated during the last decade. Their links to the models of solid-state physics [5–13] have been found, and they are based on the possibility to introduce also the spin exchange interaction in a translation-invariant form. However, the CM models

[1] Permanent address: BLTP JINR, 141980 Dubna, Moscow Region, Russia.

can be formulated in classical and quantum mechanics for any root system [14–19], and one can guess that introduction of spin exchange can be done at least for some root systems too. There were several attempts [8–10, 12] in this direction, but they were far from being universal in a way for introducing spin into the CM models.

In this paper, we consider the possibility of unifying all the previous approaches to spin Calogero–Moser models and related models of spin exchange interactions obtained by 'freezing' the canonical variables at the equilibrium points of the corresponding classical CM systems. This can be done by constructing universal Lax representations for degenerate forms of the CM potentials. There are also some indications that the corresponding models with most general elliptic potentials are also integrable [6, 13], but the construction of Lax pair in this case does not lead directly to integrability.

The organization of the paper is as follows. In section 2, the universal Lax operators for the CM models with degenerate potentials [17, 18] are briefly recapitulated. The way of introducing spin exchange in the framework of the above formalism is proposed in section 3 so as to prove the integrability of the spin CM models for all root systems. The existence of conserved quantities is guaranteed by the 'sum to zero' condition for the second Lax operator. Section 4 is devoted to the models with spin exchange operators only. The corresponding Lax operators lead in the trigonometric case and A_r root system to Haldane–Shastry model [5]. The Polychronakos model [8, 12] corresponds in this approach to the rational case with a confining q^2 potential. The final section is devoted to summary and comments.

2. Universal Lax operator for Calogero–Moser model with degenerate potential

In this section we briefly recapitulate the essence of Calogero–Moser models based on any root system Δ (applicable also to the exceptional and non-crystallographic root system) and the associated universal Lax pair formalism along with appropriate notation [16–19] and background [14, 15] for the main body of this paper. Those who are familiar with the universal Lax pair formulation may skip this section and return when necessity arises. A Calogero–Moser model is a Hamiltonian system associated with a root system Δ of rank (r), which is a set of vectors in $\mathbf{R}^r$ with its standard inner product, invariant under reflections in the hyperplane perpendicular to each vector in Δ. In other words,

$$s_\alpha(\beta) \in \Delta \qquad \forall\alpha, \beta \in \Delta \qquad s_\alpha(\beta) = \beta - (\alpha^\vee \cdot \beta)\alpha \qquad \alpha^\vee \equiv 2\alpha/|\alpha|^2. \tag{2.1}$$

The set of reflections $\{s_\alpha,\ \alpha \in \Delta\}$ generates a group G_Δ, a finite reflection group, known as the Coxeter (Weyl) group. The set of roots Δ is decomposed into a disjoint sum of the positive roots Δ_+ and negative roots Δ_-. The dynamical variables of the Calogero–Moser model are the coordinates $\{q_j\}$ and their canonically conjugate momenta $\{p_j\}$, which will be denoted by vectors in $\mathbf{R}^r$ with the standard inner product:

$$q = (q_1, \ldots, q_r) \qquad p = (p_1, \ldots, p_r) \qquad p^2 = p \cdot p = \sum_{j=1}^{r} p_j^2. \tag{2.2}$$

The Hamiltonian for classical Calogero–Moser model with a degenerate potential reads

$$\mathcal{H}_C = \frac{1}{2}p^2 + \frac{1}{2}\sum_{\rho\in\Delta_+} g_{|\rho|}^2 |\rho|^2\, V(\rho \cdot q), \tag{2.3}$$

in which the potential function V is listed in table 1. Here we have omitted the scale factor for the trigonometric (hyperbolic) potential, for simplicity. The associated universal Lax pair operators read

$$L = p \cdot \hat{H} + X \qquad X = \mathrm{i} \sum_{\rho \in \Delta_+} g_{|\rho|} \, (\rho \cdot \hat{H}) \, x(\rho \cdot q) \, \hat{s}_\rho \tag{2.4}$$

$$\tilde{M} = \frac{\mathrm{i}}{2} \sum_{\rho \in \Delta_+} g_{|\rho|} |\rho|^2 \, y(\rho \cdot q) \, \hat{s}_\rho \tag{2.5}$$

in which the functions $x(u)$ and $y(u)$ are listed in table 1. These functions are related by

$$y(u) \equiv \mathrm{d}x(u)/\mathrm{d}u \qquad V(u) = -y(u) = x^2(u) + \text{constant}. \tag{2.6}$$

The real *positive* coupling constants $g_{|\rho|}$ are defined on orbits of the corresponding reflection group, i.e. they are identical for roots in the same orbit. That is, for the simple Lie algebra cases one coupling constant $g_{|\rho|} = g$ for all roots in simply laced models and two independent coupling constants, $g_{|\rho|} = g_L$ for long roots and $g_{|\rho|} = g_S$ for short roots in non-simply laced models. The operators $\hat{H}_j$ and $\hat{s}_\rho$ obey the following commutation relations:

$$[\hat{H}_j, \hat{H}_k] = 0 \tag{2.7}$$

$$[\hat{H}_j, \hat{s}_\alpha] = \alpha_j (\alpha^\vee \cdot \hat{H}) \hat{s}_\alpha \tag{2.8}$$

$$\hat{s}_\alpha \hat{s}_\beta \hat{s}_\alpha = s_{s_\alpha}(\beta) \qquad \hat{s}_\alpha^2 = 1 \qquad \hat{s}_{-\alpha} = \hat{s}_\alpha \tag{2.9}$$

In terms of these commutation relations it is easy to show that the canonical equations of motion can be represented in an operator form:

$$\dot{q}_j = p_j \qquad \dot{p}_j = -\frac{\partial \mathcal{H}_{\mathrm{C}}}{\partial q_j} \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t} L = [L, \tilde{M}]. \tag{2.10}$$

Table 1. Functions appearing in the Hamiltonian and Lax pair.

	$V(u)$	$x(u)$	$y(u)$
Rational	$1/u^2$	$1/u$	$-1/u^2$
Trigonometric	$1/\sin^2 u$	$\cot u$	$-1/\sin^2 u$
Hyperbolic	$1/\sinh^2 u$	$\coth u$	$-1/\sinh^2 u$

Let us choose a set of D vectors $\mathcal{R}$ which form a single orbit of the reflection (Weyl) group G_Δ. It is a collection of $\mathbf{R}^r$ vectors, each is called a 'site':

$$\mathcal{R} = \left\{ \mu^{(1)}, \ldots, \mu^{(D)} \middle| \mu^{(k)} \in \mathbf{R}^r \right\}. \tag{2.11}$$

That is any site of $\mathcal{R}$ can be obtained from any other site by the action of the reflection (Weyl) group. Thus the (length)^2 of the vectors $\mu^{(k)}$ are equal:

$$\left(\mu^{(j)}\right)^2 = \left(\mu^{(k)}\right)^2 \qquad \forall \mu^{(j)}, \mu^{(k)} \in \mathcal{R}. \tag{2.12}$$

Then L and $\tilde{M}$ are $D \times D$ matrices whose elements are given by

$$(\hat{H}_j)_{\mu\nu} = \mu_j \delta_{\mu\nu} \qquad (\hat{s}_\rho)_{\mu\nu} = \delta_{\mu, s_\rho(\nu)} = \delta_{\nu, s_\rho(\mu)}. \tag{2.13}$$

The essence of the Lax pair is the following set of identities among the functions $\{x(\rho \cdot q)\}$ and $\{y(\rho \cdot q)\}$ expressed in matrix forms:

$$[X, \tilde{M}] = -\hat{H} \cdot \frac{\partial \mathcal{V}}{\partial q} \qquad \mathcal{V} = \frac{1}{2} \sum_{\rho \in \Delta_+} g_{|\rho|}^2 |\rho|^2 V(\rho \cdot q) \tag{2.14}$$

$$[p \cdot \hat{H}, \tilde{M}] = \mathrm{i} \left[-\frac{1}{2} \frac{\partial^2}{\partial q^2} X \right] \tag{2.15}$$

in which the right-hand side of (2.14) is a diagonal matrix. The matrix $\tilde{M}$ has a special property (see (2.36) of [18]):

$$\sum_{\mu\in\mathcal{R}} \tilde{M}_{\mu\nu} = \sum_{\nu\in\mathcal{R}} \tilde{M}_{\mu\nu} = -\mathrm{i}\mathcal{V}_S \qquad \mathcal{V}_S = \frac{1}{2}\sum_{\rho\in\Delta_+} g_{|\rho|}|\rho|^2\, V(\rho\cdot q) \tag{2.16}$$

in which $\mathcal{V}_S$ is independent of μ and ν. Note that $\mathcal{V}_S$ is different from $\mathcal{V}$ in (2.14), which is quadratic in the coupling $g_{|\rho|}$, whereas $\mathcal{V}_S$ is linear. We can define a new matrix M,

$$M = \tilde{M} + \mathrm{i}\mathcal{V}_S \times I \qquad I\text{: Identity operator} \tag{2.17}$$

which satisfies *sum up to zero* condition

$$\sum_{\mu\in\mathcal{R}} M_{\mu\nu} = \sum_{\nu\in\mathcal{R}} M_{\mu\nu} = 0. \tag{2.18}$$

Since the elements of the matrices X and M are numbers and $\mathcal{V}_S \times I$ commutes with X, we have from (2.14)

$$[X, M] = -\hat{H}\cdot\frac{\partial\mathcal{V}}{\partial q} \tag{2.19}$$

which is the content of the usual Lax pair.

3. Spin Calogero–Moser model with degenerate potential

Now let us define a spin Calogero–Moser model associated with a root system Δ and a set of vectors $\mathcal{R}$ forming a single orbit of the reflection (Weyl) group G_Δ. In other words $\mathcal{R}$ is the set of 'sites'. A dynamical state of the model is a wavefunction $\psi(q)$ times a vector ψ_S which takes value in the D multiple of a vector space $\mathbf{V}$;

$$\psi_S \in \overset{D}{\otimes}\mathbf{V}. \tag{3.1}$$

Each $\mathbf{V}$ is associated with site μ. In other words ψ_S can be represented by its component spin $\psi_S^{(\mu)}$ at the site μ, or $\psi_S^{(j)}$ at site j for short

$$\psi_S = \left|\psi_S^{(1)},\ldots,\psi_S^{(D)}\right\rangle. \tag{3.2}$$

Let us introduce a spin exchange operator $\hat{\mathcal{P}}_\rho$ associated with each root $\rho\in\Delta$

$$\hat{\mathcal{P}}_\rho\colon \psi_S \to \hat{\mathcal{P}}_\rho\psi_S \qquad (\hat{\mathcal{P}}_\rho\psi_S)^{(\mu)} = \psi_S^{(s_\rho(\mu))} \qquad \forall\mu\in\mathcal{R}. \tag{3.3}$$

Obviously $\{\hat{\mathcal{P}}_\rho\}$ $(\rho\in\Delta)$ satisfy the same commutation relations as $\{\hat{s}_\rho\}$:

$$\hat{\mathcal{P}}_\alpha\hat{\mathcal{P}}_\beta\hat{\mathcal{P}}_\alpha = \hat{\mathcal{P}}_{s_\alpha(\beta)} \qquad \hat{\mathcal{P}}_\alpha^2 = 1 \qquad \hat{\mathcal{P}}_{-\alpha} = \hat{\mathcal{P}}_\alpha \tag{3.4}$$

and $\hat{s}_\alpha$, $\hat{H}_j$ and $\hat{\mathcal{P}}_\beta$ commute since they act on different spaces

$$[\hat{s}_\alpha, \hat{\mathcal{P}}_\beta] = 0 = [\hat{H}_j, \hat{\mathcal{P}}_\beta]. \tag{3.5}$$

Likewise, the quantum operators $\{q_j\}$ and $\{p_k\}$ commute with $\hat{\mathcal{P}}_\rho$

$$[q_j, \hat{\mathcal{P}}_\rho] = 0 = [p_k, \hat{\mathcal{P}}_\rho] \qquad j,k = 1,\ldots,r \quad \forall\rho\in\Delta. \tag{3.6}$$

By multiplying $\hat{\mathcal{P}}_\rho$ to the functions $x(\rho\cdot q)$ and $y(\rho\cdot q)$ in X and $\tilde{M}$, we define new matrices X_S and $\tilde{M}_S$:

$$X_S = \mathrm{i}\sum_{\rho\in\Delta_+} g_{|\rho|}\,(\rho\cdot\hat{H})\,x(\rho\cdot q)\,\hat{\mathcal{P}}_\rho\hat{s}_\rho \tag{3.7}$$

$$\tilde{M}_S = \frac{\mathrm{i}}{2}\sum_{\rho\in\Delta_+} g_{|\rho|}|\rho|^2\,y(\rho\cdot q)\,\hat{\mathcal{P}}_\rho\hat{s}_\rho \tag{3.8}$$

whose elements are no longer numbers but operators now. As in the previous section we define a new matrix M_S,

$$M_S = \tilde{M}_S + \mathrm{i}\mathcal{A} \times I \tag{3.9}$$

which satisfies *sum up to zero* condition, too

$$\sum_{\mu\in\mathcal{R}} (M_S)_{\mu\nu} = \sum_{\nu\in\mathcal{R}} (M_S)_{\mu\nu} = 0. \tag{3.10}$$

The operator $\mathcal{A}$ now depends on the spin exchange operators $\{\hat{\mathcal{P}}_\rho\}$

$$\mathcal{A} = \frac{1}{2}\sum_{\rho\in\Delta_+} g_{|\rho|}|\rho|^2\, V(\rho\cdot q)\hat{\mathcal{P}}_\rho. \tag{3.11}$$

Since the commutation relations of $\{\hat{H}_j, \hat{s}_\rho\}$ and $\{\hat{H}_j, \hat{\mathbf{s}}_\rho \equiv \hat{\mathcal{P}}_\rho\hat{s}_\rho\}$ are identical we have the following main result

$$[X_S, \tilde{M}_S] = -\hat{H}\cdot\frac{\partial\mathcal{V}}{\partial q} \tag{3.12}$$

in which the right-hand side does not contain operators $\{\hat{\mathcal{P}}_\rho\}$. This is because they cancel out by the relation $\hat{\mathcal{P}}_\rho^2 = 1$. The right-hand side can be replaced by the obvious identity in quantum theory

$$-\hat{H}\cdot\frac{\partial\mathcal{V}}{\partial q} = \mathrm{i}[\mathcal{H}_C, p\cdot\hat{H}]. \tag{3.13}$$

If we rewrite $\tilde{M}_S$ in terms of M_S, we obtain

$$[X_S, M_S - \mathrm{i}\mathcal{A}] = \mathrm{i}[\mathcal{H}_C,\ p\cdot\hat{H}] \tag{3.14}$$

in which the second commutator in the left-hand side no longer vanishes. By adding (3.14) to

$$[p\cdot\hat{H}, M_S - \mathrm{i}\mathcal{A}] = \mathrm{i}\left[\frac{p^2}{2}, X_S\right] \tag{3.15}$$

we arrive at the desired equation

$$[p\cdot\hat{H} + X_S, M_S] = \mathrm{i}[\mathcal{H}_S,\ p\cdot\hat{H} + X_S] \tag{3.16}$$

$$\mathcal{H}_S \equiv \mathcal{H}_C - \mathcal{A} = \frac{1}{2}p^2 + \frac{1}{2}\sum_{\rho\in\Delta_+}|\rho|^2 g_{|\rho|}(g_{|\rho|} - \hat{\mathcal{P}}_\rho)\, V(\rho\cdot q) \tag{3.17}$$

which is a *universal* Lax equation for the spin Calogero–Moser model

$$\mathrm{i}[\mathcal{H}_S, L_S] = [L_S, M_S] \qquad L_S = p\cdot\hat{H} + X_S. \tag{3.18}$$

defined by the Hamiltonian $\mathcal{H}_S$ (3.17). That is, this applies to any spin Calogero–Moser models based on any root system Δ and a set of vectors $\mathcal{R}$ forming a single orbit of the reflection (Weyl) group G_Δ and for any degenerate potentials. From this follows

$$\mathrm{i}\left[\mathcal{H}_S, L_S^k\right] = \left[L_S^k, M_S\right] \qquad \text{or}$$

$$\mathrm{i}\left[\mathcal{H}_S, \left(L_S^k\right)_{\mu\nu}\right] = \sum_{\kappa\in\mathcal{R}}\left\{\left(L_S^k\right)_{\mu\kappa}(M_S)_{\kappa\nu} - (M_S)_{\mu\kappa}\left(L_S^k\right)_{\kappa\nu}\right\}. \tag{3.19}$$

Thanks to the sum up to zero condition of M_S (3.10) we obtain the conserved quantity as the *Total sum* (Ts) of L_S^k instead of the diagonal sum (Tr):

$$\left[\mathcal{H}_S, \mathrm{Ts}\left(L_S^k\right)\right] = 0 \qquad \mathrm{Ts}\left(L_S^k\right) \equiv \sum_{\mu,\nu\in\mathcal{R}}\left(L_S^k\right)_{\mu\nu} \qquad k = 2, \ldots. \tag{3.20}$$

This type of conserved quantity was known for the A_r spin Calogero–Moser models for the set of vector weights [7, 9]. Note that (3.15) is obtained from (2.15) by replacing X and $\tilde{M}$ by X_S and $\tilde{M}_S$.

Some remarks are in order.

1. When all the spins are the same,
$$\psi_S^{(1)} = \psi_S^{(2)} = \cdots = \psi_S^{(D)}$$
the action of the spin exchange operators becomes that of the identity operator
$$\hat{\mathcal{P}}_\rho = 1 \qquad \forall \rho \in \Delta.$$
Then the Hamiltonian $\mathcal{H}_S$ (3.17) reduces to that of the quantum Calogero–Moser models and the Lax operator L_S and M_S become identical to the universal quantum Lax pair operator derived by Bordner *et al* [18].
2. The form of the spin Calogero–Moser Hamiltonian (3.17) depends on the root system Δ only, although its actual operator contents depend on the chosen set of 'sites' $\mathcal{R}$.
3. For the A_r model with the set of vector weights, the present spin Calogero–Moser coincides with the existing one. For the other root systems the present model is completely new, to the best of our knowledge (see the remarks in the following entry). It should be emphasised that even for the A_r root system the present formulation of the spin Calogero–Moser models defines many different models corresponding to many different orbits of the symmetric group $\mathcal{S}_{r+1}$, which is the Weyl group of A_r.
4. For the A_r model with the set of vector weights the number of 'sites' is $r + 1$ which is equal to the degrees of freedom of the associated particle motion, if the A_r root system is embedded into $\mathbf{R}^{r+1}$ as is done customarily. This is a rather exceptional situation. In all the other orbits of $\mathcal{S}_{r+1}$ and for all the other root systems (except for the trivial representation), the number of sites, or the dimensions of $\mathcal{R}$, is bigger than r, the rank of Δ. For example, the set of vector weights of D_r or the set of short roots for B_r consists of $2r$ vectors, which in a conventional parametrization of the roots take the form $\mathcal{R} = \{\pm\mathbf{e}_j, j = 1, \ldots, r | \mathbf{e}_j \in \mathbf{R}^r, \mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk}\}$. Our spin Calogero–Moser models require all these $2r$ sites. There are some references in which spin Calogero–Moser models for B_r, C_r, D_r or BC_r are discussed [10–12]. In all these papers, the number of sites is equal to the rank of the root systems. These are different from the present spin Calogero–Moser models.
5. The present formulation of the spin Calogero–Moser models together with the Lax pair formulation does not require any specific structure of the 'spin' space $\mathbf{V}$ attached to each site.
6. It is well-known that for the spin $1/2$ case in the A_r model with the set of vector weights, the spin exchange operators $\{\hat{\mathcal{P}}_\rho\}$ can be expressed in terms of the local Pauli spin matrix at each site as $\hat{\mathcal{P}}_{\mathbf{e}_j-\mathbf{e}_k} = (1 + \vec{\sigma}_j \cdot \vec{\sigma}_k)/2$. For the set of vector weights of D_r or $\mathcal{R}$ being the set of short roots for B_r mentioned above, we have
$$\hat{\mathcal{P}}_{\mathbf{e}_j} = [(1 + \vec{\sigma}_j \cdot \vec{\sigma}_{-j})/2] \qquad \begin{aligned} \hat{\mathcal{P}}_{\mathbf{e}_j-\mathbf{e}_k} &= [(1 + \vec{\sigma}_j \cdot \vec{\sigma}_k)/2][(1 + \vec{\sigma}_{-j} \cdot \vec{\sigma}_{-k})/2] \\ \hat{\mathcal{P}}_{\mathbf{e}_j+\mathbf{e}_k} &= [(1 + \vec{\sigma}_j \cdot \vec{\sigma}_{-k})/2][(1 + \vec{\sigma}_{-j} \cdot \vec{\sigma}_k)/2]. \end{aligned} \tag{3.21}$$
In other words, $\hat{\mathcal{P}}_{\mathbf{e}_j+\mathbf{e}_k}$ exchanges the spins at site j and $-k$ and simultaneously the spins at $-j$ and k. Similar expressions exist for other choices of $\mathcal{R}$ and root systems for $su(2)$, $su(N)$ or other spins.

7. It is easy to verify, as in the Calogero–Moser models, that the Hamiltonian $\mathcal{H}_S$ (3.17) is obtained as the lowest member of the conserved quantities derived from the Lax pair formulation:

$$\mathcal{H}_S \propto \mathrm{Ts}\left(L_S^2\right). \tag{3.22}$$

8. The conserved quantities $\{\mathrm{Ts}(L_S^k)\}$ are essentially the same as those obtained in terms of the Dunkl [20] operators, and/or the exchange operator formalism [8]. The same remark applies to the conserved quantities of the spin exchange models to be discussed in section 4. For the quantum CM models without spin, the equivalence of the Lax pair formalism and Dunkl operator formalism was proven in [19].
9. The Yangian symmetry [21,22] for the spin CM model and spin exchange model based on any root system is an interesting challenge.
10. The commutativity of the conserved quantities obtained from the above Lax pair formulation will be discussed elsewhere.

3.1. Rational spin Calogero model

In this subsection we will define the rational spin Calogero–Moser model with quadratic confining potential, to be called the rational spin Calogero model for brevity. The Hamiltonian is given by

$$\mathcal{H}_{RS} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{1}{2}\sum_{\rho\in\Delta_+}\frac{|\rho|^2 g_{|\rho|}(g_{|\rho|} - \hat{\mathcal{P}}_\rho)}{(\rho\cdot q)^2}. \tag{3.23}$$

The construction of the Lax pair follows the same pattern as the case without the spin degrees of freedom. Since the added potential $\frac{1}{2}\omega^2 q^2$ commutes with X_S, the canonical equations of motion to be obtained from $\mathcal{H}_{RS}$ are equivalent to

$$\dot{L}_S = \mathrm{i}[\mathcal{H}_{RS}, L_S] = [L_S, M_S] - \omega^2 Q \qquad Q \equiv q\cdot\hat{H} \tag{3.24}$$

in which L_S and M_S are the Lax pair for the rational $(1/(\rho\cdot q)^2)$ potential only. Let us define

$$L_S^\pm = L_S \pm \mathrm{i}\omega Q \tag{3.25}$$

whose time evolution reads

$$\dot{L}_S^\pm = [L_S^\pm, M_S] \pm \mathrm{i}\omega L^\pm. \tag{3.26}$$

Here we have used well-known relations [17,18]

$$\dot{Q} = p\cdot\hat{H} = L_S - X_S \qquad [Q, M_S] = -X_S. \tag{3.27}$$

If we define

$$\mathcal{L}_S = L_S^+ L_S^- \tag{3.28}$$

its time evolution is Lax-like:

$$\dot{\mathcal{L}}_S = \mathrm{i}[\mathcal{H}_{RS}, \mathcal{L}_S] = [\mathcal{L}_S, M_S]. \tag{3.29}$$

Thus we obtain conserved quantities

$$\mathrm{Ts}(\mathcal{L}^k) \qquad k = 1, \ldots. \tag{3.30}$$

The lowest conserved quantity $\mathrm{Ts}(\mathcal{L})$ gives the Hamiltonian $\mathcal{H}_{RS}$ (3.23)

$$\mathrm{Ts}(\mathcal{L}) \propto \mathcal{H}_{RS} + \left(\frac{r}{2} + \sum_{\rho\in\Delta_+} g_{|\rho|}\hat{\mathcal{P}}_\rho\right) \tag{3.31}$$

plus additional terms which commute with all the spin exchange operators $\{\hat{\mathcal{P}}_\rho\}$.

4. Spin exchange model

The spin exchange model is defined for a root system Δ and a set of vectors $\mathcal{R}$ forming a single orbit of the reflection (Weyl) group G_Δ. Its dynamical state is represented by a vector ψ_S only which takes value in the D multiple of a vector space $\mathbf{V}$;

$$\psi_S \in \overset{D}{\otimes}\mathbf{V}. \tag{4.1}$$

As in the spin Calogero–Moser model case each $\mathbf{V}$ is associated with site μ. In other words ψ_S can be represented by its component spin $\psi_S^{(\mu)}$ at the site μ, or $\psi_S^{(j)}$ at site j for short

$$\psi_S = \left|\psi_S^{(1)}, \ldots, \psi_S^{(D)}\right\rangle.$$

In fact, the spin exchange model is obtained from the corresponding spin Calogero–Moser model by 'freezing' the particle degrees of freedom:

$$p = 0 \qquad q = q_0 \tag{4.2}$$

in which q_0 is an equilibrium position of the classical Calogero–Moser potential

$$\left.\frac{\partial\mathcal{V}}{\partial q}\right|_{q=q_0} = 0 \qquad \mathcal{V} = \frac{1}{2}\sum_{\rho\in\Delta_+} g_{|\rho|}^2 |\rho|^2\, V(\rho\cdot q). \tag{4.3}$$

Since the rational potential without the quadratic confining potential or the hyperbolic potential do not have any equilibrium points, this automatically selects the trigonometric potential. The rational potential with the quadratic confining potential case will be discussed in the next subsection separately. The equilibrium position q_0 for the trigonometric potential is determined uniquely in each Weyl alcove. In other words, if q_0 is an equilibrium point so is $s_\alpha(q_0)$, which defines an equally integrable model. Let us fix q_0 and define X_E and $\tilde{M}_E$ in terms of the Lax pair operators of the corresponding spin Calogero–Moser model at $q = q_0$:

$$X_E = X_S|_{q=q_0} \qquad \tilde{M}_E = \tilde{M}_S|_{q=q_0}. \tag{4.4}$$

The components of the matrices X_E and $\tilde{M}_E$ are linear combinations of the spin exchange operators $\hat{\mathcal{P}}_\rho$ and the coefficients are just numbers. They satisfy a simple matrix identity

$$[X_E, \tilde{M}_E] = 0 \tag{4.5}$$

and as before $\tilde{M}_E$ has a special property:

$$\sum_{\mu\in\mathcal{R}} (\tilde{M}_E)_{\mu\nu} = \sum_{\nu\in\mathcal{R}} (\tilde{M}_E)_{\mu\nu} = -\mathrm{i}\mathcal{A}_E \qquad \mathcal{A}_E = \frac{1}{2}\sum_{\rho\in\Delta_+} g_{|\rho|}|\rho|^2\, V(\rho\cdot q_0)\hat{\mathcal{P}}_\rho.$$

As in the previous section we define a new matrix M_E,

$$M_E = \tilde{M}_E + \mathrm{i}\mathcal{A}_E \times I$$

which satisfies the *sum up to zero* condition, too

$$\sum_{\mu\in\mathcal{R}} (M_E)_{\mu\nu} = \sum_{\nu\in\mathcal{R}} (M_E)_{\mu\nu} = 0. \tag{4.6}$$

By rewriting (4.5) in terms of M_E we arrive at the Lax representation of the spin exchange model:

$$\mathrm{i}[\mathcal{H}_E, X_E] = [X_E, M_E], \tag{4.7}$$

in which the Hamiltonian $\mathcal{H}_E$ of the spin exchange model is

$$\mathcal{H}_E = \frac{1}{2}\sum_{\rho\in\Delta_+} g_{|\rho|}|\rho|^2\, V(\rho\cdot q_0)(1 - \hat{\mathcal{P}}_\rho) = -\mathcal{A}_E + \text{constant}. \tag{4.8}$$

The added constant simply shifts the ground state energy. The Lax pair supplies the conserved quantities as the *Total sum* (Ts) of X_E^k:

$$\left[\mathcal{H}_E, \mathrm{Ts}\left(X_E^k\right)\right] = 0 \qquad \mathrm{Ts}\left(X_E^k\right) \equiv \sum_{\mu,\nu\in\mathcal{R}} \left(X_E^k\right)_{\mu\nu} \qquad k = 3, \ldots. \tag{4.9}$$

It is interesting to note that the first two members $\mathrm{Ts}(X_E^1)$ and $\mathrm{Ts}(X_E^2)$ are trivial, in contrast to the spin Calogero–Moser case.

Some remarks are in order.

1. As in the spin Calogero–Moser model, the form of the spin exchange model Hamiltonian $\mathcal{H}_E$ (4.8) depends on the root system Δ only, although its actual operator contents depend on the chosen set of 'sites' $\mathcal{R}$. These many models corresponding to various orbits (or sets of sites), sharing the same set of conserved quantities, can be considered to constitute an integrable *hierarchy* belonging to the root system Δ.
2. It should be remarked that the q_0 is the equilibrium point *not* of the function appearing in the Hamiltonian $\mathcal{H}_E$ (4.8) which is linear in the coupling constants $g_{|\rho|}$ but that of the potential of the *classical* Calogero–Moser Hamiltonian $\mathcal{H}_C$ (2.3) which is quadratic in the coupling constants. This difference is meaningful only for the models based on non-simply laced root systems.
3. It should be emphasised that the 'coordinates' q or rather q_0 are just a set of numbers rather than dynamical variables. Thus, in contrast to the conventional approach [5,8], the notion of 'position exchange operator' is not used in our approach.
4. For the A_r model, q_0 can be chosen to be 'equidistant':
$$q_0 = \pi(1, 2, \ldots, r, r+1)/(r+1), \tag{4.10}$$
thanks to the well-known trigonometric identity
$$\sum_{k\neq j}^{r+1} \frac{\cos[\pi(j-k)/(r+1)]}{\sin^3[\pi(j-k)/(r+1)]} = 0.$$
The Haldane–Shastry model [5], i.e. the A_r spin exchange model with $\mathcal{R}$ being the set of vector weights, has been understood quite well because of this simplifying feature.
5. The equidistance of q_0 for A_r seems rather fortuitous. As remarked above, any transposition of the above q_0 (4.10) provides an equally integrable spin exchange model, but the equidistance property is lost. As for D_r ($r \geqslant 4$), we have not been able to find equidistant q_0. For BC_r model, equidistant q_0 can be achieved for certain ratios of the coupling constants. For the following parametrization of the potential [4, 16],
$$\mathcal{V} = \sum_{j<k}^{r} \left[\frac{g_M^2}{\sin^2(q_j - q_k)} + \frac{g_M^2}{\sin^2(q_j + q_k)}\right] + \sum_{j=1}^{r} \frac{g_S(g_S + 2g_L)}{2\sin^2(q_j)} + \sum_{j=1}^{r} \frac{2g_L^2}{\sin^2(2q_j)} \tag{4.11}$$
one obtains equidistant equilibrium positions:
$$q_0 = \pi(1, 3, \ldots, 2r-1)/4r \qquad \text{for} \quad g_L/g_M = 1/2, \quad g_S = 0 \tag{4.12}$$
$$q_0 = \pi(1, 2, \ldots, r)/2(r+1) \qquad \text{for} \quad g_L/g_M = 3/2, \quad g_S = 0 \tag{4.13}$$
$$q_0 = \pi(1, 2, \ldots, r)/(2r+1) \qquad \text{for} \quad g_L/g_M = 1/2, \quad g_S/g_M = 1. \tag{4.14}$$
These cases were discussed in some detail by Bernard *et al* [11].
6. Note that the present derivation of the spin exchange model and its Lax pair does not adopt the strong coupling limit.
7. For most general elliptic potentials, the Lax pair can be constructed in a usual manner [14]. But the second Lax operator does not satisfy the 'sum to zero' condition, hence the integrability of these models is not yet established.

4.1. Rational spin exchange model

The above formulation fails to give integrable spin exchange model with rational potential. This can be remedied by adding a harmonic confining potential [8,12] which creates equilibrium points in each Weyl chamber. Here we derive the Lax operator formalism for these models. Let us start with the Lax pair for the rational Calogero–Moser models and for the time being keep the value of q unspecified. We have as in (2.14)

$$[X, \tilde{M}] = -\hat{H} \cdot \frac{\partial \mathcal{V}}{\partial q}$$

and after multiplying $\hat{\mathcal{P}}_\rho$ to functions $x(\rho \cdot q)$ and $y(\rho \cdot q)$, we obtain (3.12)

$$[X_S, \tilde{M}_S] = -\hat{H} \cdot \frac{\partial \mathcal{V}}{\partial q}. \tag{4.15}$$

The diagonal matrix Q (3.19) satisfies the relation (3.27)

$$[Q, \tilde{M}_S] = -X_S. \tag{4.16}$$

If we define two new matrices $X_S^\pm$

$$X_S^\pm = X_S \pm \mathrm{i}\omega Q \tag{4.17}$$

they satisfy simple commutation relations thanks to (4.15) and (4.16)

$$[X_S^\pm, \tilde{M}_S] = \mp\mathrm{i}\omega X_S^\pm - \hat{H} \cdot \frac{\partial \mathcal{V}_{RC}}{\partial q} \tag{4.18}$$

in which $\mathcal{V}_{RC}$ is the potential of the classical rational Calogero–Moser model with harmonic confining potential

$$\mathcal{V}_{RC} = \frac{1}{2}\omega^2 q^2 + \frac{1}{2}\sum_{\rho\in\Delta_+} \frac{g_{|\rho|}^2|\rho|^2}{(\rho \cdot q)^2}. \tag{4.19}$$

Now we choose q_0 as an equilibrium point of $\mathcal{V}_{RC}$ and define

$$X_{RE}^\pm = X_S^\pm|_{q=q_0} \qquad \tilde{M}_{RE} = \tilde{M}_S|_{q=q_0} \qquad \left.\frac{\partial \mathcal{V}_{RC}}{\partial q}\right|_{q=q_0} = 0. \tag{4.20}$$

Thus we arrive at

$$\left[X_{RE}^+ X_{RE}^-, \tilde{M}_{RE}\right] = X_{RE}^+\left[X_{RE}^-, \tilde{M}_{RE}\right] + \left[X_{RE}^+, \tilde{M}_{RE}\right]X_{RE}^- = 0. \tag{4.21}$$

We define M_{RE} by

$$M_{RE} = \tilde{M}_{RE} + \mathrm{i}\mathcal{A}_{RE} \times I \tag{4.22}$$

$$\mathcal{A}_{RE} = \frac{1}{2}\sum_{\rho\in\Delta_+} \frac{g_{|\rho|}|\rho|^2\hat{\mathcal{P}}_\rho}{(\rho \cdot q_0)^2} \tag{4.23}$$

so that M_{RE} satisfies the *sum to zero* condition

$$\sum_{\mu\in\mathcal{R}}(M_{RE})_{\mu\nu} = \sum_{\nu\in\mathcal{R}}(M_{RE})_{\mu\nu} = 0. \tag{4.24}$$

Then (4.21) can be rewritten as a Lax representation for the rational spin exchange model

$$\mathrm{i}[\mathcal{H}_{RE}, X_{RE}^+ X_{RE}^-] = [X_{RE}^+ X_{RE}^-, M_{RE}] \tag{4.25}$$

in which the rational spin exchange Hamiltonian $\mathcal{H}_{RE}$ is defined by

$$\mathcal{H}_{RE} = \frac{1}{2}\sum_{\rho\in\Delta_+} \frac{g_{|\rho|}|\rho|^2}{(\rho \cdot q_0)^2}(1 - \hat{\mathcal{P}}_\rho) = -\mathcal{A}_{RE} + \text{constant}. \tag{4.26}$$

The conserved quantities are obtained as the *Total sum* of $\left(X^{+}_{RE}X^{-}_{RE}\right)^{k}$

$$\left[\mathcal{H}_{RE}, \mathrm{Ts}\left(\left(X^{+}_{RE}X^{-}_{RE}\right)^{k}\right)\right] = 0 \qquad k = 1, \ldots . \tag{4.27}$$

It is interesting to note that the above Hamiltonian $\mathcal{H}_{RE}$ depends on the harmonic confining potential $\frac{1}{2}\omega^2 q^2$ only through the value q_0.

5. Summary and comments

We have shown that the integrability of spin Calogero–Moser model and the spin exchange model with degenerate potential and based on any root system is a direct consequence of the integrability of the corresponding classical Calogero–Moser system. For a given root system Δ there are many integrable spin Calogero–Moser models and the spin exchange models corresponding to many choices of $\mathcal{R}$'s which are the orbits of the reflection group. These define physically different models sharing the same exchange features.

After completion of the present work, we came across [23] which discusses the integrability of spin BC_r model with harmonic confining potential, or 'spin Inozemtsev model' [4] in terms of the Dunkl operator formalism [20].

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